A KINETIC ANALYSIS OF COUETTE PLASMA FLOW IN AN ELECTRIC FIELD

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Steady flow of a completely ionized plasma between parallel plates moving in their own plane in the presence of an electric field is examined. The distribution functions of the ions and electrons are derived from the kinetic Boltzmann equations supplemented by equations for the electric field. The solution is constructed by means of a variation of the method of moments; at the same time, it is assumed that momentum is transferred only by ions and heat is transferred by electrons. The analysis takes into account near collisions between particles for an arbitrary degree of rarefaction of the plasma. An example of calculation of the principal characteristics of the flow is given.

The boundary problems connected with the presence of solid surfaces in the flow are of far less practical importance in plasma dynamics than in gasdynamics. This is because a plasma can exist only at very high temperatures which destroy the majority of materials to a greater or lesser extent. As a rule, in order to contain a plasma within definite bounds, strong magnetic fields are utilized, not solid walls. Nevertheless, consideration of problems with bounding surfaces is still of definite interest in the case of plasma. An approximate solution of one of the simplest problems of this kind is given below.

1. Let a completely ionized plasma fill the space between two infinite parallel impermeable planes, one of which moves to the right and the other to the left at constant velocities U/2 (Fig. 1). The temperatures of the plates are also constant, even though they differ in the general case; without loss of generality, we can consider that the temperature of the upper plate is higher than that of the lower one $T_u > T_d$. The entire system is within some external electric field, whose intensity vector lies in the xy-plane. The plates themselves are not charged and are dielectrics.

The distribution function of the ions F_i and the electrons F_e satisfy the Boltzmann equations

$$\frac{\partial F_i}{\partial t} + (\mathbf{c}\cdot\nabla) F_i + \frac{Ze}{m_i}\sum_{j=1}^{3} E_{sj}\frac{\partial F_i}{\partial c_j} = \Lambda F_i,$$

$$\frac{\partial F_e}{\partial t} + (\mathbf{c}\cdot\nabla) F_e - \frac{e}{m_e}\sum_{j=1}^{3} E_{sj}\frac{\partial F_e}{\partial c_j} = \Lambda F_e. \qquad (1.1)$$

Here e is the unit charge, m_i or m_e the mass of a particle, $\mathbf{E}_s$ the vector of the total intensity of the electric field, Z is the multiplicity of the charge on the ion, and the right sides of the equations include integral operators characterizing the effect of collisions of particles of a given kind among themselves and with particles having the opposite charge. In the absence of a magnetic field, $\mathbf{E}_s$ can be represented as the sum of two terms

$$\mathbf{E}_s = \mathbf{E}_0 + \mathbf{E}, \qquad (1.2)$$

the first of which corresponds to the external electric field, while the second characterizes the potential field set up by the space charge and determined by Poisson's equation

$$\nabla\mathbf{E} = 4\pi e(Zn_i - n_e). \qquad (1.3)$$

The symbols n_i and n_e denote the number densities of the particles.

In accordance with the conditions of the given problem, the left sides of Eqs. (1.1) can be simplified somewhat. Thus, by virtue of the steady-state nature of the processes under consideration, the time derivatives vanish, and by virtue of the geometry of the problem, it is necessary to equate the derivatives with respect to x, z, and c_z to zero.

For an approximate solution of the problem, we shall go from the Boltzmann equations to moment equations obtained from (1.1) by multiplying by some function of the molecular velocities $\varphi(c_x, c_y, c_z)$ and integrating over the entire range of variation of the latter. In our case, we obtain moment equations of the form

$$\frac{d}{dy}\int c_y\varphi F_\alpha\, dV =$$

$$= \frac{Z_\alpha e E_{sx}}{m_\alpha}\int\frac{\partial\varphi}{\partial c_x}F_\alpha\, dV + \frac{Z_\alpha e E_{sy}}{m_\alpha}\int\frac{\partial\varphi}{\partial c_y}F_\alpha\, dV + \Lambda_\alpha\varphi$$

$$\left(\Lambda_\alpha\varphi = \int\varphi\Lambda F_\alpha\, dV, \quad \int dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dc_x\, dc_y\, dc_z\right). \qquad (1.4)$$

The subscript α in Eq. (1.4) may take values i or e; and $Z_i = Z$, $Z_e = -1$.

In addition to equations like (1.4), we need Eq. (1.3), which, in this case, takes the form

$$\frac{dE_y}{dy} = 4\pi e(Zn_i - n_e), \qquad (1.5)$$

and the condition of irrotationality of the electric field, written in the form

$$\frac{d}{dy}(E_{0x} + E_x) = 0. \qquad (1.6)$$

To satisfy condition (1.6), we can set without loss of generality

$$E_x \equiv 0, \qquad E_{0x} = \text{const} = E_0.$$

We shall make use of the method proposed by Lees and Liu [1], representing each of the required functions by two Maxwellian distribution functions:

$$F_\alpha = F_{\alpha_1} = \qquad (1.7)$$

$$= n_{\alpha_1}\left(\frac{m_\alpha}{2\pi kT_{\alpha_1}}\right)^{3/2}\exp\left[-m_\alpha\frac{(c_x - v_{x\alpha_1})^2 + c_y^2 + c_z^2}{2kT_{\alpha_1}}\right] \text{ when } c_y < 0,$$

$$F_\alpha = F_{\alpha_2} = \qquad (1.7)\ \text{(cont'd)}$$
$$= n_{\alpha_2}\left(\frac{m_\alpha}{2\pi kT_{\alpha_2}}\right)^{3/2} \exp\left[-m_\alpha \frac{(c_x - v_{x\alpha_2})^2 + c_y^2 + c_z^2}{2kT_{\alpha_2}}\right] \text{ when } c_y > 0.$$

The quantities n_{α_1}, n_{α_2}, T_{α_1}, T_{α_2}, $v_{x\alpha_1}$, and $v_{x\alpha_2}$ in expressions (1.7), functions of the y-coordinate, should be determined with the aid of moment equations like (1.4) with six functions φ_j ($j = 1, 2, \ldots, 6$) for each kind of particle.

When a gas consisting of neutral particles is investigated by a similar method, representing the distribution function in accordance with (1.7) ensures an exact solution of the boundary problem for the Boltzmann equation in the limiting case of free-molecule flow. When studying a plasma, we can no longer completely satisfy Eqs. (1.1) this way, even when there are no collisions between particles. However, the object of this method is not to construct an exact solution of the Boltzmann equations, and on going to the moment equations, representation in the form of (1.7) is very convenient from the standpoint of formulating boundary conditions and evaluating the integrals.

2. The selection of the functions φ_j is to a certain extent arbitrary. As the first four functions, we take the quantities m_α, $m_\alpha c_x$, $m_\alpha c_y$, $\frac{1}{2}m_\alpha(c_x^2 + c_y^2 + c_z^2)$. These functions are additive invariants of the collisions; thus, when they are substituted in equations like (1.4), the integrals $\Lambda_\alpha\varphi$ turn out to be nonzero only due to exchange of momentum and energy between particles of type α and oppositely charged particles; the integral $\Lambda_\alpha\varphi_1$ which vanishes due to absence of mass transfer between particles is an exception. On the basis of the remarks made here, one can set, in accordance with definition,

$$\Lambda_\alpha\varphi_1 = 0, \qquad \Lambda_\alpha\varphi_2 \equiv \int m_\alpha c_x \Lambda F_\alpha\, dV = R_{x\alpha},$$
$$\Lambda_\alpha\varphi_3 = R_{y\alpha}, \quad \Lambda_\alpha\varphi_4 = Q_\alpha, \qquad (2.1)$$

where R_α denotes the force caused by collisions of particles of the type α with oppositely charged particles, and Q_α the energy dissipated as a result of such collisions.

Two more functions φ_j are needed, for which we take $\varphi_5 = m_\alpha c_x c_y$, $\varphi_6 = \frac{1}{2}m_\alpha c_y(c_x^2 + c_y^2 + c_z^2)$. In the general case, the collision integrals $\Lambda_\alpha\varphi_5$ and $\Lambda_\alpha\varphi_6$ cannot be found analytically, and the result of their numerical determination depends on the law of interaction between particles. However, in these integrals, it is not difficult to isolate the terms corresponding to formulas (2.1). Thus, introducing the determination of the thermal particle velocity by the formula $w = c - v_\alpha$, we obtain

$$\Lambda_\alpha\varphi_5 = \int m_\alpha c_x c_y \Lambda F_\alpha\, dV =$$
$$= v_{x\alpha}\int m_\alpha c_y \Lambda F_\alpha\, dV + \int m_\alpha w_x w_y \Lambda F_\alpha\, dV =$$
$$= v_{x\alpha} R_{y\alpha} + \int m_\alpha w_x w_y \Lambda F_\alpha\, dV, \qquad (2.2)$$

$$\Lambda_\alpha\varphi_6 = \frac{1}{2}\int m_\alpha c_y (c_x^2 + c_y^2 + c_z^2)\Lambda F_\alpha\, dV = \frac{1}{2} v_{x\alpha}^2 R_{y\alpha} +$$
$$+ v_{x\alpha}\int m_\alpha w_x w_y \Lambda F_\alpha\, dV + \frac{1}{2}\int m_\alpha w_y (w_x^2 + w_y^2 + w_z^2)\Lambda F_\alpha\, dV.$$

The integral terms in the right sides of (2.2) play a less important role than the preceding ones and can be evaluated approximately by replacing the integral operator ΔF_α by a simplified model. Following the example of reference [2], we take

$$\Lambda F_\alpha \approx \frac{p_\alpha}{\mu_\alpha}(F_\alpha^{(0)} - F_\alpha). \qquad (2.3)$$

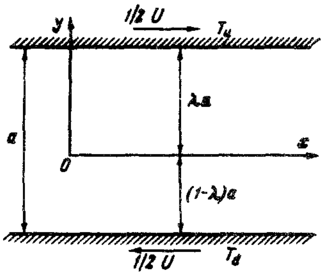

Fig. 1

Here p_α denotes the partial pressure, μ the viscosity coefficient of particles of the given type, corresponding to the second approximation of the Chapman-Enskog theory [3], $F_\alpha^{(0)}$ is the Maxwellian distribution function.

Substituting (2.3) in the integrand of formulas (2.2) and taking into consideration that

$$\int w_x w_y F_\alpha^{(0)}\, dV = \int w_y w^2 F_\alpha^{(0)}\, dV = 0$$

and also recalling the definition of the heat flux vector and the stress tensor

$$q_\alpha = \frac{1}{2} m_\alpha \int w w^2 F_\alpha\, dV, \qquad P_{nk} = m_\alpha \int w_n w_k F_\alpha\, dV,$$

we obtain

$$\int m_\alpha w_x w_y \Lambda F_\alpha\, dV \approx -\frac{p_\alpha}{\mu_\alpha} P_{xy\alpha},$$
$$\frac{1}{2}\int m_\alpha w_y w^2 \Lambda F_\alpha\, dV \approx -\frac{p_\alpha}{\mu_\alpha} q_{y\alpha}.$$

Formulas (2.2) can now be rewritten in the form

$$\Lambda_\alpha\varphi_5 = v_{x\alpha} R_{y\alpha} - \frac{p_\alpha}{\mu_\alpha} P_{xy\alpha},$$
$$\Lambda_\alpha\varphi_6 = \frac{1}{2} v_{x\alpha}^2 R_{y\alpha} - \frac{p_\alpha}{\mu_\alpha}(v_{x\alpha} P_{xy\alpha} + q_{y\alpha}). \qquad (2.4)$$

The viscosity coefficients for charged particles interacting by the Coulomb law can be expressed as in [3]

$$\mu_\alpha = \frac{5}{8A_2(2)}\left(\frac{m_\alpha kT_\alpha}{\pi}\right)^{1/2}\left(\frac{2kT_\alpha}{Z_\alpha^2 e^2}\right)^2,$$

$$A_2(2) = 2\left[\ln(1 + v_{01}^2) - \frac{v_{01}^2}{1 + v_{01}^2}\right],$$

$$v_{01} = \frac{4dkT_\alpha}{Z_\alpha^2 e^2} \qquad (Z_i = Z,\ Z_e = -1). \qquad (2.5)$$

Here d is the average distance between particles.

We introduce the notation

$$I_{5\alpha} = m_\alpha \int c_x c_y^2 F_\alpha \, dV, \qquad I_{6\alpha} = \frac{m_\alpha}{2} \int c_y^2 c^2 F_\alpha \, dV. \quad (2.6)$$

The moment equations like (1.4) for the functions φ_j (j = 1, 2, . . . , 6) selected can now be written in the form

$$\frac{d}{dy}(n_\alpha v_{y\alpha}) = 0, \qquad \frac{dP_{xy\alpha}}{dy} = Z_\alpha n_\alpha e E_0 + R_{x\alpha},$$

$$\frac{dP_{yy\alpha}}{dy} = Z_\alpha n_\alpha e E_y + R_{y\alpha},$$

$$\frac{d}{dy}(v_{x\alpha} P_{xy\alpha} + q_{y\alpha}) = Z_\alpha n_\alpha e E_0 v_{x\alpha} + Q_\alpha,$$

$$\frac{dI_{5\alpha}}{dy} = Z_\alpha n_\alpha e E_y v_{x\alpha} + R_{y\alpha} v_{x\alpha} - \frac{p_\alpha}{\mu_\alpha} P_{xy\alpha},$$

$$\frac{dI_{6\alpha}}{dy} = Z_\alpha \frac{eE_0}{m_\alpha} P_{xy\alpha} + \frac{3Z_\alpha e E_y}{m_\alpha} P_{yy\alpha} +$$

$$+ \frac{1}{2} v_{x\alpha}^2 R_{y\alpha} - \frac{p_\alpha}{\mu_\alpha}(v_{x\alpha} P_{xy\alpha} + q_{y\alpha}). \quad (2.7)$$

With the aid of the second equation of (2.7), the fourth equation is transformed into the more convenient form

$$P_{xy\alpha} \frac{dv_{x\alpha}}{dy} + \frac{dq_{y\alpha}}{dy} = -R_{x\alpha} v_{x\alpha} + Q_\alpha. \quad (2.8)$$

If we assume that the process of near collision of an electron and an ion possesses the same properties as the process of elastic collision of smooth bodies, then it is not difficult to obtain the relationship

$$\mathbf{R}_i = -\mathbf{R}_e,$$

$$Q_i + Q_e = -(\mathbf{R}_i \mathbf{v}_i + \mathbf{R}_e \mathbf{v}_e) = R_{xe}(v_{xi} - v_{xe}). \quad (2.9)$$

The dependence of the quantities in Eqs. (2.7) on the conventional "flow parameters" corresponds to representation of a two-flow function by formulas (1.7) and takes the form

$$n_\alpha = \frac{1}{2}(n_{\alpha_1} + n_{\alpha_2}), \qquad v_{x_\alpha} = \frac{n_{\alpha_1} v_{x\alpha_1} + n_{\alpha_2} v_{x\alpha_2}}{n_{\alpha_1} + n_{\alpha_2}},$$

$$v_{y\alpha} = \left(\frac{2}{\pi m_\alpha}\right)^{1/2} \frac{n_{\alpha_2}\sqrt{kT_{\alpha_2}} - n_{\alpha_1}\sqrt{kT_{\alpha_1}}}{n_{\alpha_1} + n_{\alpha_2}},$$

$$P_{xx\alpha} = \frac{k}{2}(n_{\alpha_1} T_{\alpha_1} + n_{\alpha_2} T_{\alpha_2}) +$$

$$+ m_\alpha \left[\frac{1}{2}(n_{\alpha_1} v_{x\alpha_1}^2 + n_{\alpha_2} v_{x\alpha_2}^2) - n_\alpha v_{x\alpha}^2\right],$$

$$P_{yy\alpha} = P_{zz\alpha} = \frac{k}{2}(n_{\alpha_1} T_{\alpha_1} + n_{\alpha_2} T_{\alpha_2}),$$

$$P_{xy\alpha} = \left(\frac{m_\alpha k T_{\alpha_1}}{2\pi}\right)^{1/2} n_{\alpha_1}(v_{x\alpha_2} - v_{x\alpha_1}),$$

$$q_{y\alpha} = \frac{n_{\alpha_1}}{2}\left(\frac{m_\alpha k T_{\alpha_1}}{2\pi}\right)^{1/2}\left[4\frac{k}{m_\alpha}(T_{\alpha_2} - T_{\alpha_1}) +\right.$$

$$\left. + v_{x\alpha_2}^2 - v_{x\alpha_1}^2 - 2v_{x\alpha}(v_{x\alpha_2} - v_{x\alpha_1})\right],$$

$$I_{5\alpha} = \frac{k}{2}(n_{\alpha_1} T_{\alpha_1} v_{x\alpha_1} + n_{\alpha_2} T_{\alpha_2} v_{x\alpha_2}).$$

$$I_{6\alpha} = \frac{n_{\alpha_1}}{4}\sqrt{kT_{\alpha_1}}\left[\frac{5}{m_\alpha}(kT_{\alpha_1})^{3/2} +\right.$$

$$\left. + \frac{5}{m_\alpha}(kT_{\alpha_2})^{3/2} + v_{x\alpha_1}^2\sqrt{kT_{\alpha_1}} + v_{x\alpha_2}^2\sqrt{kT_{\alpha_2}}\right],$$

$$p_\alpha = n_\alpha k T_\alpha = \frac{1}{3}(P_{xx\alpha} + P_{yy\alpha} + P_{zz\alpha}). \quad (2.10)$$

3. Considering the steady-state motion of plasma under the conditions of the given problem, it is natural to assume that the temperatures of the ions T_i and the electrons T_e, and also the densities n_i and n_e, are of the same order of magnitude. If we also assume that the macroscopic velocities of both components are not of an order higher than the corresponding average thermal velocities, then one can obtain important estimates for the displacement stress $P_{xy\alpha}$ and the heat flux $q_{y\alpha}$ from an examination of formulas (2.10). Indeed, it can be seen from these formulas that, other conditions being equal,

$$\frac{P_{xye}}{P_{xyi}} \approx \frac{q_{yi}}{q_{ye}} \approx \left(\frac{m_e}{m_i}\right)^{1/2}.$$

Bearing in mind the smallness of the quantity $\sqrt{m_e / m_i}$, we conclude that it is possible to neglect the quantities P_{xye} and q_{yi} as compared with P_{xyi} and q_{ye}, respectively.

In order to simplify the formulation of the boundary conditions when solving the problem of Couette plasma flow, we shall consider that the particle charge is not changed when the particles are reflected from a surface. Moreover, in accordance with the assumptions made above, we assume that when ions are reflected, there is total accommodation of the tangential component of the momentum and when electrons are reflected, there is total accommodation of the thermal energy to conditions on the surface. In other words, the macroscopic velocity of reflected ions at the surface is equal to the velocity of motion of the plate, and the temperature of the reflected electrons at the point of reflection is equal to the temperature of the plate.

As appears from the above, the true distribution of the macroscopic velocity of the electrons is not important in determining the total displacement stress in the flow. Thus, for example, one can assume that $v_{xe_1} = v_{xi_1}$, $v_{xe_2} = v_{xi_2}$; as we shall see later, this means that $v_{xe} = v_{xi} = v_x$. Turning to the problem of the temperature of the ions and bearing in mind the stationarity of the processes under investigation, we shall consider the plasma to be in equilibrium, that is, we shall set $T_{i_1} = T_{e_1} = T_1$ and $T_{i_2} = T_{e_2} = T_2$; as we shall see later, these two relationships are equivalent to the condition $T_i = T_e$.

We shall consider the density of the ions to be proportional to the density of the electrons, that is, $n_{i_1} = Dn_{e_1}$, $n_{i_2} = Dn_{e_2}$, $n_i = Dn_e$. The proportionality factor D depends in this case on the degree of rarefaction of the plasma and on the multiplicity of the charge on the ions Z. The total density of the plasma will be expressed as

$$n = n_i + n_e = (1 + D)\, n_e, \quad (3.1)$$

with analogous expressions for n_1 and n_2. Taking these assumptions into account, the boundary conditions of the problem take the form

$$v_{x_1} = {}^1\!/_2 U, \quad T_1 = T_u = \chi T_d \text{ when } y = \lambda a, \qquad (3.2)$$
$$v_{x_2} = -{}^1\!/_2 U, \quad T_2 = T_d, \quad n_2 = n_d \text{ when } y = (\lambda - 1) a .$$

The last condition in regard to density should have been formulated in a somewhat different way; essentially, we should have given here the total mass of some vertical column of the substance participating in the motion. However, since this mass was not previously known, this integral condition can be replaced by a simpler one, as in (3.2); however, the exact value of the constant will be determined after solving the entire problem. It is necessary to add to the conditions (3.2) the condition of impermeability of both surfaces $v_{y\alpha} = 0$ when $y = {}^1\!/_2 a\,[(2\lambda - 1) \pm 1]$. We note that, integrating the first equation of (2.7), taking this condition into account, we obtain $v_{y\alpha} \equiv 0$ over the whole flow field; the satisfaction of the last identity is assumed in deriving certain equations.

As a result of these assumptions, the number of unknown functions in Eqs. (2.7) to be solved jointly with Eq. (1.5) is decreased. Making use of this fact and recalling relation (2.9), we eliminate the quantities $R_{x\alpha}$ and $R_{y\alpha}$ from the system (2.7). Thus, by adding termwise the second equations of this system, written for α = i and α = e, we get

$$\frac{dP_{xyi}}{dy} = (Zn_i - n_e)\, eE_0 = \frac{E_0}{4\pi}\frac{dE_y}{dy},$$

since $P_{xye} \equiv 0$ by hypothesis. In a like manner, writing $P_{yyi} + P_{yye} = P_{yy}$, we obtain from the third equations of the same system

$$\frac{dP_{yy}}{dy} = \frac{1}{4\pi} E_y \frac{dE_y}{dy} .$$

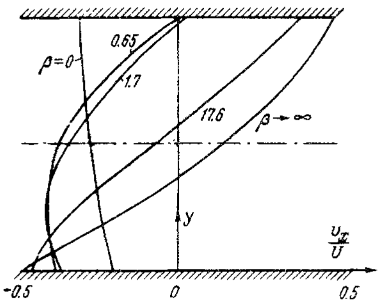

Fig. 2

The fifth and sixth equations of system (2.7) should, generally speaking, be considered separately for α = = i and α = e. This means, in particular, that the ratios $n_{i_1}/n_{e_1} = D_1$ and $n_{i_2}/n_{e_2} = D_2$ can be variables not equal to each other. However, if we consider the plasma as a whole and take into account that there cannot be large deviations from quasi-linearity in it, we can approximately set $D_1 = D_2 = D$ = const, thus obtaining the relation (3.1). However, it is obvious from the foregoing that along with acceptance of the relation (3.1), it is necessary to reject consideration of the fifth and sixth equations of system (2.7) for each of the components separately. Therefore, we adopt the notation $I_{5i} + I_{5e} = I_5$ and $m_e I_{6i} + m_e I_{6e} = I_6$, and introduce equations for determining the functions I_5 and I_6; the equations obtained will be given somewhat later for the case of small ion Mach numbers.

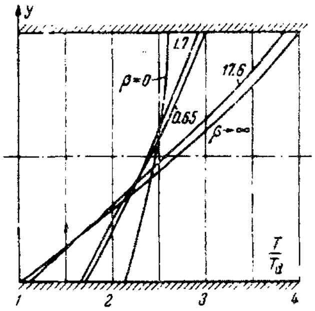

Fig. 3

Turning to formula (2.5), we express the viscosity coefficient as

$$\mu_\alpha = B\sqrt{m_\alpha}\, T^{5/2}, \qquad (3.3)$$

where the constant coefficient B is assumed not to depend on the type of particle; this condition is strictly satisfied only for a quasilinear plasma with Z = 1. In many cases, it is permissible to linearize formula (3.3) with the aid of a method widely applied in boundary layer theory (see, for example, [4]). If $\mu_{\alpha d}$ is the value of the viscosity coefficient for particles of the given kind near the lower plate, then, in place of (3.3), we can accept the approximate relation

$$\frac{\mu_\alpha}{\mu_{\alpha d}} \approx C\,\frac{T}{T_d}, \qquad C = \left(\frac{T_u}{T_d}\right)^{3/4} = \chi^{3/4}. \qquad (3.4)$$

It is more convenient to continue further analysis in dimensionless variables introduced by the formulas

$$v_x = v_x{}^\circ U, \quad n = n^\circ n_d, \quad T = T^\circ T_d,$$
$$P_{xy\alpha} = P_{xy}{}^\circ \frac{n_\alpha}{n}\left(\frac{m_\alpha k T_d}{2\pi}\right)^{1/2} n_d U, \qquad P_{yy} = P_{yy}{}^\circ \frac{n_d k T_d}{2},$$
$$q_{y\alpha} = q_y{}^\circ \frac{n_\alpha}{n}\frac{n_d}{\sqrt{2\pi m_\alpha}}(kT_d)^{3/2},$$
$$I_{5\alpha} = I_5{}^\circ \frac{n_\alpha}{n} n_d k T_d U, \qquad I_{6\alpha} = I_6{}^\circ \frac{n_\alpha}{n}\frac{n_d}{m_\alpha} k^2 T_d{}^2,$$
$$E_y = E_y{}^\circ E_* \equiv E_y{}^\circ E_0 \frac{D+1}{D}\left(\frac{\varkappa k T_d}{m_i U^2}\right)^{1/2}, \qquad y = y^\circ \nu a, \quad (3.5)$$

where ν is a positive coefficient still undetermined (it will be shown later that ν corresponds to the average spatial value of dimensionless density). The transformed equations will contain a number of dimensionless parameters, including

$$M_\alpha = U\left(\frac{m_\alpha}{\varkappa k T_d}\right)^{1/2}, \qquad R_\alpha = \frac{U a m_\alpha n_d}{\mu_{\alpha d}}, \qquad \gamma = \frac{eE_* a}{kT_d},$$
$$\varepsilon = \frac{4\pi e n_d a}{E_*}, \qquad \beta = \frac{16}{15\sqrt{2\pi\varkappa}}\frac{1+D^2}{(1+D)^2}\frac{R_\alpha}{CM_\alpha}. \qquad (3.6)$$

If the viscosity coefficient is expressed by formula (3.3), the parameter β will not depend on the type of particle.

Let us restrict ourselves to the case of small ion Mach numbers in which terms of the order of M_i^2 can be neglected as compared with terms of the order of unity. Moreover, let D = O (1), γ = O (1). With these restrictions, Eqs. (2.7) and (1.5) are transformed to the form (the superscript is omitted for dimensionless quantities)

$$n_2\sqrt{T_2}=n_1\sqrt{T_1}\qquad \frac{dP_{xy}}{dy}=\left(\frac{2\pi}{\varkappa}\right)^{1/2}\frac{\Upsilon}{\varepsilon}\frac{dE_y}{dy},$$
$$\frac{dP_{yy}}{dy}=2\frac{\Upsilon}{\varepsilon}E_y\frac{dE_y}{dy},\qquad \frac{dq_y}{dy}=0,$$
$$\frac{dI_5}{dy}=\frac{\Upsilon}{\varepsilon}v_xE_y\frac{dE_y}{dy}-\frac{15}{16}n\nu\beta P_{xy},\tag{3.7}$$
$$\frac{dI_6}{dy}=3\frac{\Upsilon}{\varepsilon}TE_y\frac{dE_y}{dy}-\frac{15}{8}n\nu\beta q_y,\qquad \frac{dE_y}{dy}=n\nu\varepsilon\frac{ZD-1}{1+D}.$$

If we substitute into Eqs. (3.7) the expressions (2.10) reduced to dimensionless form by formulas (3.5) and simplified for the case $M_1^2 \ll 1$, then it becomes clear that the system obtained decomposes into two; that is, determination of the densities, temperatures, and electric field intensity can be achieved independently of the determination of velocities.

4. We now introduce a new independent variable

$$\eta=\int_0^y n\,dy.\tag{4.1}$$

Now, the first, third, fourth, sixth, and seventh equations of system (3.7) take the form ($\sigma \equiv T$)

$$n_2\sigma_2=n_1\sigma_1\tag{4.2}$$

$$\frac{d}{d\eta}[n_1\sigma_1(\sigma_1+\sigma_2)]=2\frac{\Upsilon}{\varepsilon}E_y\frac{dE_y}{d\eta},\tag{4.3}$$

$$n_1\sigma_1(\sigma_2^2-\sigma_1^2)=\alpha_2=\text{const},\tag{4.4}$$

$$\frac{d}{d\eta}[n_1\sigma_1(\sigma_1^3+\sigma_2^3)]=$$
$$=\frac{12}{5}\frac{\Upsilon}{\varepsilon}\frac{n_1\sigma_1(\sigma_1+\sigma_2)}{n_1+n_2}E_y\frac{dE_y}{d\eta}-\frac{3}{2}\nu\beta\alpha_2,\tag{4.5}$$

$$\frac{dE_y}{d\eta}=\nu\varepsilon b\qquad\left(b=\frac{ZD-1}{1+D}\right).\tag{4.6}$$

Integrating Eq. (4.6), we obtain

$$E_y=\nu\varepsilon b\eta+E',\tag{4.7}$$

where E' is the constant of integration determined from the boundary conditions for E_y. After this, Eq. (4.3) yields

$$n_1\sigma_1(\sigma_1+\sigma_2)=$$
$$=\nu\gamma b(\nu\varepsilon b\eta^2+2E'\eta)+\alpha_3\equiv\alpha_3+g(\eta)\tag{4.8}$$

With the purpose of transforming Eq. (4.5) to a more convenient form, we note that, taking (4.2) into consideration, the average temperature of the plasma is expressed as

$$T=\frac{n_1\sigma_1(\sigma_1+\sigma_2)}{n_1+n_2}=\sigma_1\sigma_2.\tag{4.9}$$

In addition, we obtain from Eqs. (4.4) and (4.5)

$$\sigma_2=\sigma_1+\frac{\alpha_2}{\alpha_3+g(\eta)}.\tag{4.10}$$

Making use of (4.9), also (4.7), (4.8), and (4.10), we can represent Eq. (4.5) in the form (primed quantities are derivatives with respect to η)

$$(\alpha_3+g)T'-\frac{1}{5}(\alpha_3+g)'T-$$
$$-\frac{\alpha_2^2(\alpha_3+g)'}{(\alpha_3+g)^2}+\frac{3}{2}\nu\beta\alpha_2=0.\tag{4.11}$$

Integration of the linear equation (4.11) yields

$$T=-\frac{5}{11}\alpha_2^2(\alpha_3+g)^{-2}-$$
$$-\frac{3}{2}\nu\beta\alpha_2(\alpha_3+g)^{1/5}\Gamma(\eta)+\alpha_4(\alpha_3+g)^{1/5},\tag{4.12}$$
$$\Gamma(\eta)=\int_{\eta_0}^{\eta}[\alpha_3+g(\eta)]^{-6/5}d\eta,$$

where α_4 is a new constant of integration.

It is not difficult to find σ_1 and σ_2 (the sign preceding the radical is selected from the condition $\sigma_1 + \sigma_2 \geq$ ≥ 0) from (4.12) with the aid of (4.9) and (4.10):

$$\sigma_1=[\alpha_3+g(\eta)]^{-1}\left[\Omega(\eta)-\frac{\alpha_2}{2}\right],$$
$$\sigma_2=[\alpha_3+g(\eta)]^{-1}\left[\Omega(\eta)+\frac{\alpha_2}{2}\right],\tag{4.13}$$
$$\Omega(\eta)=\sqrt{-{}^9/_{44}\alpha_2^2-{}^3/_2\nu\beta(\alpha_3+g)^{11/5}\Gamma(\eta)+\alpha_4(\alpha_3+g)^{11/5}}.$$

After this, one can find the values n_1, n_2 and the average dimensionless density of the plasma n with the aid of (4.2) and (4.8). Moreover, the density can be expressed as follows from Eqs. (4.8) and (4.9):

$$n=\frac{\alpha_3+g(\eta)}{2T(\eta)},\tag{4.14}$$

where $T(\eta)$ is determined according to (4.12).

The constants α_2, α_3, and α_4 included in expressions (4.12), (4.13), and (4.14) can be found with the aid of the group of boundary conditions (3.2) associated with temperature and pressure. In this case, the coefficients λ and ν [refer to Fig. 1 and the last of the formulas (3.5)] can always be chosen so that the upper plate corresponds to the value $\eta = 1/2$ and the lower one to the value $\eta = -1/2$. Thus, the boundary conditions used at this stage are of the form

$$\sigma_1({}^1/_2)=\sqrt{\chi},\qquad \sigma_2(-{}^1/_2)=1,\qquad n_2(-{}^1/_2)=1.\tag{4.15}$$

In formula (4.12), we set $\eta_0 = -1/2$ and introduce the notation

$$\alpha_3+g(-{}^1/_2)=\alpha_3',\qquad \alpha_2=\alpha_2'\alpha_3',\qquad \alpha_4=\alpha_4'(\alpha_3')^{-1/5},$$
$$\Delta g=g({}^1/_2)-g(-{}^1/_2).$$

With the aid of formulas (4.8), (4.10), and (4.12), also the second and third boundary conditions of (4.15), we obtain

$$\alpha_2'=2-\alpha_3',\ \alpha_4'=$$
$$=1-\alpha_2'+{}^5/_{11}\alpha_2'^2=\alpha_3'-1+{}^5/_{11}(\alpha_3'-2)^2.\tag{4.16}$$

Further, after substituting the first of the conditions of (4.15) into formula (4.12), we obtain the equation

$$\frac{5}{11}\left(\frac{\alpha_3'}{\alpha_3'+\Delta g}\right)^2 \alpha_2'^2 + \frac{3}{2}\nu\beta\alpha_2' \frac{\alpha_3'}{\alpha_3'+\Delta g}(\alpha_3'+\Delta g)^{3/2}\Gamma(1/2) -$$

$$-\alpha_4'\left(\frac{\alpha_3'}{\alpha_3'+\Delta g}\right)^{-1/2} + \chi + \sqrt{\chi}\alpha_2' \frac{\alpha_3'}{\alpha_3'+\Delta g} = 0. \quad (4.17)$$

The quantities α_2' and α_4' are expressed through α_3' by formulas (4.16) so that there remains one unknown α_3' in Eq. (4.17). In the general case, this equation will be transcendental since α_2' figures implicitly in the expression $\Gamma(1/2)$ and the function $\Gamma(\eta)$ [refer to (4.12)] can be represented in elementary form only for a certain form $g(\eta)$; thus, in order to solve Eq. (4.17), it is essential to apply some approximate method.

5. After determining the quantities σ_1, σ_2, n_1, and n_2, it is necessary to go on to determining v_{x_1} and v_{x_2}, turning to the second and fifth equations of system (3.7) for this purpose. With the aid of the same transformations as those used in obtaining Eqs. (4.2)–(4.6), the equations under consideration can be represented in the form

$$n_1\sigma_1(v_{x_2} - v_{x_1}) = \sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1 \quad (5.1)$$

$$\frac{d}{d\eta}[n_1\sigma_1(\sigma_1 v_{x_1} + \sigma_2 v_{x_2})] =$$

$$= 2\nu\gamma b E_y v_x - \frac{15}{8}\nu\beta(\sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1). \quad (5.2)$$

It should be borne in mind that the average velocity of the ions is expressed as

$$v_x = \frac{n_1 v_{x_1} + n_2 v_{x_2}}{n_1 + n_2}.$$

Taking this formula into consideration, Eq. (5.2) can be reduced to the form

$$dv_x/d\eta =$$

$$= -\frac{\alpha_2}{2}\frac{d}{d\eta}\left[\frac{\sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1}{(\alpha_3+g)^2}\right] - \frac{\alpha_2\sqrt{2\pi/\varkappa}\,\nu\gamma b}{2(\alpha_3+g)^2} -$$

$$-\frac{15}{8}\nu b\frac{\sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1}{\alpha_3+g}. \quad (5.3)$$

Equation (5.3) can be immediately integrated to yield

$$v_x = -\alpha_2\frac{\sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1}{2(\alpha_3+g)^2} -$$

$$-\frac{1}{2}\alpha_2\left(\frac{2\pi}{\varkappa}\right)^{1/2}\nu\gamma b\int_{-1/2}^{\eta}\frac{d\eta}{(\alpha_3+g)^2} -$$

$$-\frac{15}{8}\nu\beta\int_{-1/2}^{\eta}\frac{\sqrt{2\pi/\varkappa}\,\nu\gamma b\eta + \alpha_1}{\alpha_3+g}d\eta + \alpha_5. \quad (5.4)$$

With the assumptions we have made, the dimensionless form for writing the boundary conditions for the velocities is as follows:

$$v_{x_1}(1/2) = 1/2, \qquad v_{x_2}(-1/2) = -1/2.$$

If we make use of relation (5.1) and the results obtained previously, the boundary conditions for the average velocity can be represented in the form

$$v_x\left(\frac{1}{2}\right) = \frac{1}{2} + \sqrt{\chi}\,\frac{\sqrt{\pi/2\varkappa}\,\nu\gamma b + \alpha_2}{\alpha_3 + g(1/2)},$$

$$v_x\left(-\frac{1}{2}\right) = -\frac{1}{2} + \frac{\sqrt{\pi/2\varkappa}\,\nu\gamma b - \alpha_2}{\alpha_3 + g(-1/2)} \quad (5.5)$$

The constants α_1 and α_2 figuring in formula (5.4) and the quantities still unknown can be found without difficulty with the aid of the boundary conditions (5.5).

6. With the aid of (4.14), the transition to the dimensionless physical variable y is performed using the formula

$$y = \int_0^{\eta}\frac{d\eta}{n(\eta)} = 2\int_0^{\eta}\frac{T(\eta)}{\alpha_3 + g(\eta)}d\eta, \quad (6.1)$$

where $T(\eta)$ is expressed according to (4.13). In the general case, the integral in the right side of formula (6.1) should be determined numerically. It is not difficult to see that the previously introduced coefficients, ν and λ, will now be expressed as

$$\nu = \left[\int_{-1/2}^{1/2}\frac{d\eta}{n(\eta)}\right]^{-1}, \qquad \lambda = \nu\int_0^{1/2}\frac{d\eta}{n(\eta)}. \quad (6.2)$$

We shall now compute the average value of the number density $\langle n\rangle$ of the plasma by the width of the flow between the plates:

$$\frac{\langle n\rangle}{n_d} = \left(\int_{(\lambda-1)/\nu}^{\lambda/\nu} n\,dy\right)\left(\int_{(\lambda-1)/\nu}^{\lambda/\nu} dy\right)^{-1} = \nu\int_{-1/2}^{1/2} d\eta = \nu. \quad (6.3)$$

Clearly, the quantity ν has a definite physical sense and is proportional to the average spatial value of the density. As for the quantity λ, it characterizes the inhomogeneity of the density distribution; if $\lambda < 1/2$, then the average value of the density in the upper part of the flow region ($y > 0$) is greater than in the lower part, but if $\lambda > 1/2$, then the increase in density is in the vicinity of the lower plate.

It should be noted that in the formulas we derived for the density, temperature, and velocity, some of the dimensionless parameters appear with the factor ν, and this quantity, as can be seen, is itself determined from the solution of the problem. On the other hand, it was shown in formulating the boundary conditions (3.2), that the quantity n_d was also previously unknown and should be determined essentially on the basis of the assignment of the average density over the width of the flow $\langle n\rangle$. If the quantity ν is found, then as can be seen from formula (6.3),

$$n_d = \langle n\rangle\,\nu^{-1}.$$

Consequently, the factor ν can be excluded from all results if $\langle n\rangle$, not n_d is accepted as the characteristic value of the density in formulas (3.5) and (3.6). In this case, however, the boundary condition for the density of reflected particles n_2 would be changed.

With this, we can conclude the description of the general scheme for solving the problem of Couette plasma flow. The case $\chi = T_u/T_d = 4$ was considered as an example of the application of this scheme.

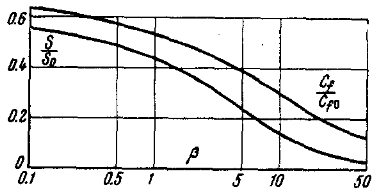

Fig. 4

As was pointed out, the average velocity of the ions is considered to be considerably lower than the corresponding speed of sound, and the parameter β [the principal rarefaction parameter, refer to (3.6)] can take any values from 0 to ∞. In addition, when the degree of rarefaction is changed, then, generally speaking, the other dimensionless parameters will change; for example, γ, ε, and b. In order to establish just how they will change, we assume that the quantity β changes only due to density when the values of the temperature and the velocities of the plates are held constant. If, in this case, the value of E^* also remains constant, this implies that one can consider that γ = const.

On the other hand, we have

$$\varepsilon\gamma = \frac{4\pi n_d a^2 e^2}{kT_d} = \frac{a^2}{r_D^2}.$$

Here r_D is the Debye-Hückel radius. Under the assumption made above, the value of a/r_D is proportional to $\sqrt{\beta}$ and if we take account of the constancy of the parameter γ, it is necessary to consider that $\varepsilon \sim \beta$.

As for the quantity b, it should depend on a/r_D and vanish when $a/r_D \to \infty$. Consequently, one can set, for example,

$$b = b_0 e^{-\theta\sqrt{\beta}},$$

where b_0 and θ are certain given constants.

As noted, the parameters $\beta_1 = \nu\beta$, $\gamma_1 = \nu\gamma$, and $\varepsilon_1 = \nu\varepsilon$ take the place of β, γ, and ε in the equations we need. For the sake of simplicity in carrying out the computations, we have set

$$\gamma_1 = \text{const} = 1,\ \varepsilon_1 = \beta_1$$
$$b = -{}^3/_5 e^{-\sqrt{\beta_1}},\ E' = -{}^5/_3.$$

Figures 2 and 3 show the velocity and temperature profiles obtained under the above-mentioned conditions for several values of the rarefaction parameter β. As might be expected, in the limiting case $\beta \to \infty$, the profiles coincide with the corresponding profiles for ordinary Couette flow with compressibility taken into consideration.

Making use of the dimensional notation of the tangential stress P_{xyi}, the local coefficient of friction can be determined as

$$C_f = -\frac{2P_{xyi}}{m_i n_d U^2} = -\frac{1+b}{(1+Z)M_i}\left(\frac{2\pi}{\varkappa}\right)^{1/2}\left[\left(\frac{2\pi}{\varkappa}\right)^{1/2}\nu\gamma b\eta + \alpha_1\right]. \quad (6.4)$$

Unlike hydrodynamic Couette flow, the coefficient of friction is found to vary over the width of the flow region. When comparing various flow regimes, the Mach number is considered to be a constant, thus by using the subscript 0 to denote the values corresponding to the case of collisionless flow when $\beta = 0$ for conditions on the lower plate ($\eta = -1/2$), we obtain

$$\frac{C_f}{C_{f0}} = \frac{1+b}{1+b_0}\,\frac{\sqrt{\pi/2\varkappa}\,\nu\gamma b - \alpha_1}{\sqrt{\pi/2\varkappa}\,\nu\gamma b_0 - \alpha_{10}}.$$

The graph showing the ratio C_f/C_{f_0} as a function of β is shown in Fig. 4. The Stanton number has been taken here as the heat transfer characteristic.

$$S = \frac{q_{ye}}{m_e n_a c_{pe} U(T_d - T_u)} =$$
$$= \frac{\varkappa-1}{\varkappa}\left(\frac{m_i}{m_e}\right)^{1/2}\frac{Z-b}{Z+1}\,\frac{\alpha_2}{\sqrt{2\pi\varkappa}\,M_i(1-\chi)}.$$

A graph of the variation of the quantity S/S_0 is also given in Fig. 4.

The curves of the variation of the coefficients of friction and heat transfer in Couette plasma flow do not contain any essential singularities, and at high values of β behave just like the corresponding curves for a neutral gas.

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